HYPERBOLAS

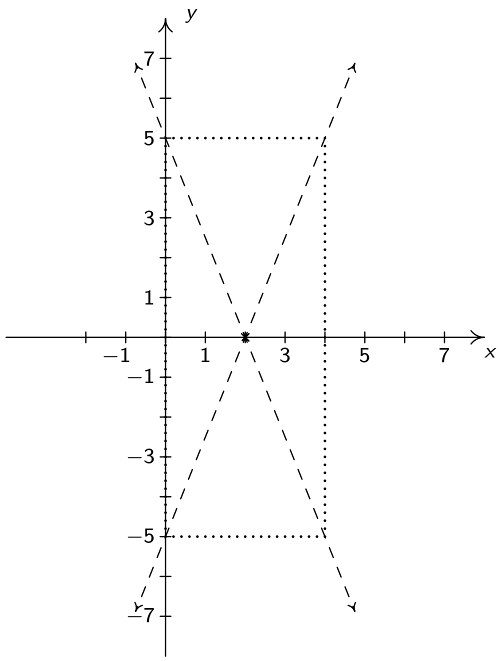
EXAMPLE:

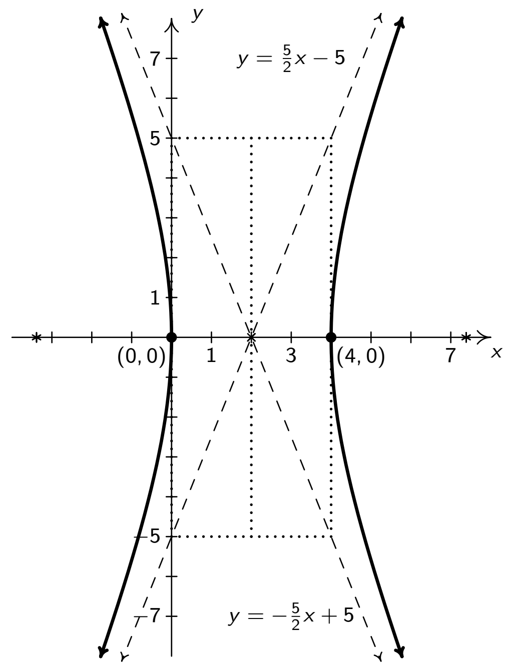
1. (a) We rewrite $25(x-2)^2 - 4y^2 = 100$, to put it into standard form:

$$25(x-2)^2 - 4y^2 = 100 \to \frac{(x-2)^2}{4} - \frac{(y-0)^2}{25} = 1 \leftrightarrow \frac{(x-2)^2}{(2)^2} - \frac{(y-0)^2}{(5)^2} = 1.$$

We identify $h = 2$ and $k = 0$, so the hyperbola is centered at $(2, 0)$. We also see $a = 2$, and $b = 5$, which means we move 2 units to the left and to the right of the center and 5 units up and down from the center to arrive at points on the guide rectangle: $(0, 0)$, $(4, 0)$, $(2, 5)$, and $(2, -5)$. Since slant asymptotes pass through the center of the hyperbola as well as the corners of the rectangle, we get the set-up as drawn below on the left.

Since the y^2 term is being subtracted from the x^2 term, the branches of the hyperbola open to the left and right so the transverse axis lies along the x-axis and the conjugate axis lies along the vertical line $x = 2$. We get the graph below.





The graph of $25(x-2)^2 - 4y^2 = 100$.

Since the vertices of the hyperbola are where the hyperbola intersects the transverse axis, we get that the vertices are $(0, 0)$ and $(4, 0)$. To find the foci, we need $c = \sqrt{a^2 + b^2} = \sqrt{4 + 25} = \sqrt{29}$. Since the foci lie on the transverse axis, we move $\sqrt{29}$ units to the left and right of $(2, 0)$ to arrive at $(2 - \sqrt{29}, 0)$ (approximately $(-3.39, 0)$) and $(2 + \sqrt{29}, 0)$ (approximately $(7.39, 0)$).

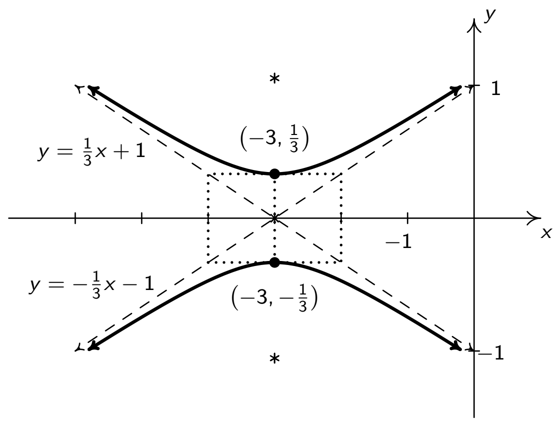
Lastly, to determine the equations of the asymptotes, recall that the asymptotes pass through the center of the hyperbola, $(2, 0)$, as well as the corners of guide rectangle. As such, they have slopes of $\pm\frac{b}{a} = \pm\frac{5}{2}$. Feeding this information into the point-slope equation of a line, we get $y - 0 = \pm\frac{5}{2}(x - 2)$, so the asymptotes are $y = \frac{5}{2}x - 5$ and $y = -\frac{5}{2}x + 5$.

(b) Once again, we need to put the given equation into standard form.

$$\begin{aligned}
9y^2 - x^2 - 6x &= 10 & \\
9y^2 - 1\left(x^2 + 6x\right) &= 10 & \text{Factor out leading coefficient of } x^2. \\
9y^2 - 1\left(x^2 + 6x + \underline{9}\right) &= 10 + (-1)\left(\underline{9}\right) & \text{Complete the Square in } x. \\
9y^2 - (x+3)^2 &= 1 & \text{Factor.} \\
\frac{y^2}{\frac{1}{9}} - \frac{(x+3)^2}{1} &= 1 & \text{Rewrite.} \\
\frac{(y-0)^2}{\left(\frac{1}{3}\right)^2} - \frac{(x-(-3))^2}{(1)^2} &= 1 & \text{Rewrite.}
\end{aligned}$$

We identify $h = -3$ and $k = 0$ so the center is $(-3, 0)$. We also see so $a = 1$, and $b = \frac{1}{3}$ which means that we move 1 unit to the left and to the right of the center and $\frac{1}{3}$ units up and down from the center to arrive at points on the guide rectangle: $(-4, 0)$, $(-2, 0)$, $\left(-3, \frac{1}{3}\right)$ and $\left(-3, -\frac{1}{3}\right)$.

Since the x^2 term is being subtracted from the y^2 term, we know the branches of the hyperbola open upwards and downwards. This means the transverse axis lies along the vertical line $x = -3$ and the conjugate axis lies along the x-axis. As a result, we get the vertices are $\left(-3, \frac{1}{3}\right)$ and $\left(-3, -\frac{1}{3}\right)$.



The graph of $9y^2 - x^2 - 6x = 10$.

To find the foci, we use $c = \sqrt{a^2 + b^2} = \sqrt{\frac{1}{9} + 1} = \frac{\sqrt{10}}{3}$. Since the foci lie on the transverse axis, we move $\frac{\sqrt{10}}{3}$ units above and below $(-3, 0)$ to arrive at $\left(-3, \frac{\sqrt{10}}{3}\right)$ and $\left(-3, -\frac{\sqrt{10}}{3}\right)$.

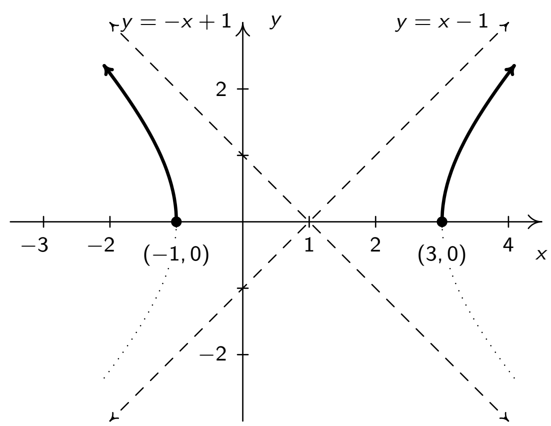
To determine the asymptotes, we use the fact the asymptotes pass through the center of the hyperbola, $(-3, 0)$, as well as the corners of guide rectangle, so they have slopes of $\pm\frac{b}{a} = \pm\frac{1}{3}$. Once again we use the point-slope equation of a line to get the two asymptotes $y = \frac{1}{3}x + 1$ and $y = -\frac{1}{3}x - 1$.

2. Graphing $f(x) = \sqrt{x^2 - 2x - 3}$ amounts to graphing the equation $y = \sqrt{x^2 - 2x - 3}$. In order to use the tools we've learned in this chapter, we first square both sides to get a quadratic equation in two variables:

$$\begin{array}{rcll}
y & = & \sqrt{x^2 - 2x - 3} & \\
y^2 & = & \left(\sqrt{x^2 - 2x - 3}\right)^2 & \text{Square both sides.} \\
y^2 & = & x^2 - 2x - 3 & \\
y^2 - x^2 + 2x & = & -3 & \\
y^2 - 1\left(x^2 - 2x\right) & = & -3 & \text{Factor out leading coefficient of } x^2. \\
y^2 - 1\left(x^2 - 2x + \underline{1}\right) & = & -3 + (-1)\left(\underline{1}\right) & \text{Complete the square in } x. \\
y^2 - (x-1)^2 & = & -4 & \text{Factor.} \\
-\dfrac{y^2}{4} + \dfrac{(x-1)^2}{4} & = & 1 & \text{Divide through by } -4. \\
\dfrac{(x-1)^2}{(2)^2} - \dfrac{(y-0)^2}{(2)^2} & = & 1 & \text{Rewrite.}
\end{array}$$

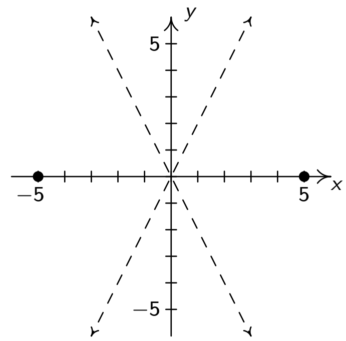
We identify $h = 1$ and $k = 0$ so the center is $(1, 0)$. We have $a = b = 2$, which means we move 2 units to the left, to the right, up and down from the center to find points on the guide rectangle: $(-1, 0)$, $(3, 0)$, $(1, -2)$ and $(1, 2)$. Of these four points, the vertices are $(-1, 0)$ and $(3, 0)$ since the hyperbola opens to the left and to the right. As usual, we the guide rectangle helps us sketch the hyperbola along with its slant asymptotes, which we find are $y - 0 = \pm(x - 1)$ or $y = x - 1$ and $y = -x + 1$.

We know since f is a function, the graph of f cannot be the *entire* hyperbola, otherwise the graph would fail the vertical line test. Since, by definition, $\sqrt{x^2 - 2x - 3} \geq 0$, we know $f(x) \geq 0$. Hence the graph of f is the *upper* half of the hyperbola, as shown below.



The graph of $f(x) = \sqrt{x^2 - 2x - 3}$.

3. (a) We plot the data given to us below.



We see the branches of the hyperbola open to the left and to the right. This means the our answer will take the form of

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Since the center is the midpoint of the vertices, we see the center is $(0,0)$, so $h = k = 0$. Moreover, since the vertices are exactly 5 units from the center, we know $a = 5$ so $a^2 = 25$. All that remains to find is the value of b^2.

Recall that the slopes of the asymptotes are $\pm\frac{b}{a}$. Since $a = 5$ and the slope of the line $y = 2x$ is 2, we have that $\frac{b}{5} = 2$, so $b = 10$. Hence, $b^2 = 100$. Our final answer is

$$\frac{x^2}{25} - \frac{y^2}{100} = 1.$$

(b) From what we are given on the graph, the equation of the hyperbola takes the form of

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1.$$

The vertices appear to be $(3,1)$ and $(3,-3)$ whose midpoint gives us the center as $(3,-1)$. Hence, $h = 3$ and $k = -1$. Moreover, since the vertices are 2 units above and below the center, we know $b = 2$ so $b^2 = 4$. All that remains is for us to find the value of a^2.

Since we are given two additional points, $(0,3)$ and $(0,-5)$, we choose one of them, $(0,3)$ to find a^2 and use the other, $(0,-5)$ to partially check our answer.

At this stage, we know the equation of the hyperbola is

$$\frac{(y+1)^2}{4} - \frac{(x-3)^2}{a^2} = 1,$$

so substituting $x = 0$ and $y = 3$ into this equation, we get $\frac{16}{4} - \frac{9}{a^2} = 1$ so $a^2 = 3$. Hence, our final answer is

$$\frac{(y+1)^2}{4} - \frac{(x-3)^2}{3} = 1.$$